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A FRACTURE-ENERGY-BASED ELASTO-SOFTENING-PLASTIC CONSTITUTIVE MODEL FOR JOINTS OF GEOMATERIALS *

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Abstract: *On the basis of plasticity and fracture mechanics for quasi-brittle materials, this article presented a constitutive model for gradual softening behavior of joints of geomaterials. Corresponding numerical tests are carried out at the local level. Characteristics of the model proposed are* 1) *plastic softening and dilatancy behavior are directly related to the fracture process of joint, and much less material and model parameters are required compared with those proposed by references*; 2) *the process of decohesion coupled with frictional sliding at both micro-scale and macro-scale is described.*

Key words: interface crack; quasi-brittle fracture; joint element; dilatancy; non-associated plasticity

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Introduction

For many years, as a tool dealing with the mechanical discontinuity properties of joints of geomaterials such as concrete and rock mass, joint element has got wide applications in geo-engineering[1]. Most of these models of joint element were established in the framework of conventional plasticity and has nothing to do with fracture mechanics. Usually in the calculation with joint element, Coulomb type yielding criterion is adopted to distinguish inelastic state from elastic state. In recent years, as an extension of cohesive crack model[2], numerous researchers had proposed their fracture-mechanics-based new joint element constitutive models[3,4]. The principal novelties of these new joint elements are that interface behavior was treated with advantage as a pseudo-material problems. Wear and dilatancy on the interface were considered. Mixed-mode decohesion was treated as coupled softening in tension and shear. The problem was then solved in the framework of conventional plasticity. Principal characters of this kind of models are its linear softening laws for interface variables. However, for these new joint

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constitutive models, rather many additional parameters are required.

This study is aimed at the formulation of a piecewise fracture-energy-based linear elasto-plastic constitutive model for joints of geomaterials in the framework of deformation theory of plasticity[5], and further, extend it to incremental formulation. Corresponding numerical tests are carried out. The characteristics of the model proposed in this article are 1) plastic softening and dilatancy behaviour are directly related to the failure process of the joint, and much less parameters are required compared with those proposed by references; 2) the process of decohesion coupled with frictional sliding is described by this model.

1 Mathematical Formulation

For generality of the formulation, definitions of variables at the joint surface are given in 3-dimension and they can be reduced to 2-dimensional cases. Definitions of interface variables at a joint surface are given as follows: $\boldsymbol{p}$ represents the traction vector on the joint surface, $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ the displacement vector on two sides of the joint respectively, and $\boldsymbol{w}$ represents vector of joint opening, i.e., displacement discontinuity across the joint

$$\boldsymbol{p} = \{p_n \quad \boldsymbol{p}_t\}^{\mathrm{T}}, \quad \boldsymbol{w} = \boldsymbol{u}^+ - \boldsymbol{u}^- = \{w_n \quad \boldsymbol{w}_t\}, \tag{1}$$

where p_n is normal traction component, $\boldsymbol{p}_t$ is tangential traction vector, w_n is normal component of joint opening, $\boldsymbol{w}_t$ is tangential vector of joint opening. In problems considered here, deformation is assumed as small, in the sense that equilibrium relations are not influenced by configuration changes. And also it is assumed the mechanical process is an isothermal process.

1.1 Piecewise linear elasto-softening-plastic constitutive model for variables at joint surface

1.1.1 General formulation of piecewise linear deformation theory of plasticity for joint surface variables

It is assumed that the vector of displacement discontinuity across the joint consists of two parts: elastic part $\boldsymbol{w}^{\mathrm{e}}$ and plastic part $\boldsymbol{w}^{\mathrm{p}}$.

$$\boldsymbol{w} = \boldsymbol{w}^{\mathrm{e}} + \boldsymbol{w}^{\mathrm{p}}. \tag{2}$$

The elastic part of displacement discontinuity can be calculated with the elastic stiffness matrix $\boldsymbol{K}$ of the joint material and the traction vector $\boldsymbol{p}$:

$$\boldsymbol{w}^{\mathrm{e}} = \boldsymbol{K}^{-1}\boldsymbol{p}. \tag{3}$$

The elastic stiffness matrix $\boldsymbol{K}$ can be expressed as

$$\boldsymbol{K} = \begin{bmatrix} K_n & 0 \\ 0 & K_t \end{bmatrix}, \tag{4}$$

where K_n and K_t are stiffness modulus in normal direction and tangential direction to the joint plane respectively. If an joint layer of thickness t is assumed, then there is

$$K_n = \frac{E}{t}, \quad K_t = \frac{G}{t}, \tag{5}$$

where E and G are elastic Young's modulus and shear modulus respectively.

The plastic part of displacement discontinuity can be calculated with deformation theory of plasticity, and the associated flow law or nonassociated flow laws in the sense of conventional plasticity, that is,

$$w^{p} = V\lambda, \tag{6}$$

where vector V defines the direction of the vector of plastic displacement discontinuity, λ defines the vector of plastic multipliers. If associated flow law is adopted, then

$$V = \frac{\partial \varphi}{\partial p} = N, \tag{7}$$

where φ is the vector of piecewise linear yielding surfaces, and therefore N is a constant direction matrix of the piecewise linear plastic yielding surfaces.

If nonassociated flow law is adopted, then

$$V = \frac{\partial \varphi'}{\partial p} \neq N, \tag{8}$$

where φ' is the vector of piecewise linear plastic potential function.

The total-quantity-form elasto-plastic relationship between the vector of displacement discontinuity w across the joint surface and corresponding known traction vector p at the joint surface can be then written as

$$w = K^{-1}p + V\lambda. \tag{9}$$

The matrix-form of piecewise linear plastic yielding criterion for given traction vector p are

$$\varphi = N^{T}p - H\lambda - Y \leqslant 0, \tag{10}$$

where H is the displacement-softening matrix, Y is the vector of material strength parameters in the direction related to each yielding surface. Following the fundamental assumption of classical deformation theory of plasticity, Eq. (10) means that co-ordinates of a traction point on the consequent yielding surfaces is uniquely determined by the total quantities of internal variable λ, and is independent of loading path.

1.1.2 Formulation of cohesive laws at the joint surface: fracture mechanics

The process of decohesion coupled with frictional sliding is one of the major concerns of this study. Two kinds of crack opening mechanism are assumed to be consistent in the failure process: mode Ⅰ crack opening and mode Ⅱ crack. The linear softening law is adopted in description of cohesive behavior for both tensional and shear cracks. As shown in Fig. 1, the softening of strength parameters is defined on the space of displacement discontinuity across the joint surface and the traction, i.e., the spaces of $w_n - p_n$. Here it is defined that H_n denotes displacement softening modulus in direction normal to the joint surface; H_t denotes displacement softening modulus in tangential direction to the joint.

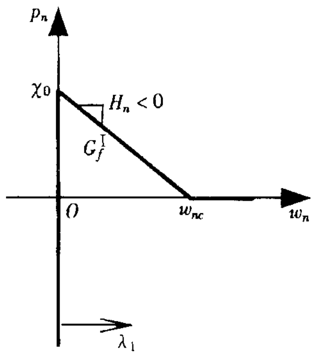

Fig. 1 Cohesive law in normal direction

H_n and H_t are related to mode Ⅰ fracture energy of G_f^{I}, and mode Ⅱ, G_f^{II}, through the following equations (see Fig. 1):

$$G_f^{\mathrm{I}} = \frac{1}{2}\chi_0 w_{nc}, \quad \chi_0 = -H_n w_{nc}. \tag{11}$$

From Eq.(11), it can be obtained

$$H_n = \begin{cases} -\dfrac{\chi_0^2}{2G_f^{\mathrm{I}}} & (\text{if } w_n < w_{nc}), \\ 0 & (\text{if } w_n \geqslant w_{nc}), \end{cases} \tag{12}$$

where w_{nc} is the Critical Opening Displacement value of mode Ⅰ crack. By analogy to Eq.(12), it can be obtained

$$H_t = \begin{cases} -\dfrac{c_0^2}{2G_f^{\mathrm{II}}} & (\text{if } w_t < w_{tc}), \\ 0 & (\text{if } w_t \geqslant w_{tc}), \end{cases} \tag{13}$$

where w_{tc} is the Critical Opening Displacement value of model Ⅱ crack.

The description of the dissipative process related to joint failure is formulated by using internal variable λ_1 for mode Ⅰ crack and λ_2 for mode Ⅱ crack. λ_3 for mode Ⅱ crack are also used for the purpose to distinguish the negative shear loading from the case of negative shear loading.

1.1.3 Coupling between softening piecewise linear plasticity and fracture mechanics of cohesive crack

The so-called fracture-mechanics-based softening plasticity means that, the cohesive law of the joint variables determines the softening properties of plasticity. For the case of piecewise linear plasticity adopted here, the softening behaviour related to strength vector $\boldsymbol{Y}$ in Eq.(10) is dependent on the internal variables λ_i, $i = 1,3$ at the joint surfaces.

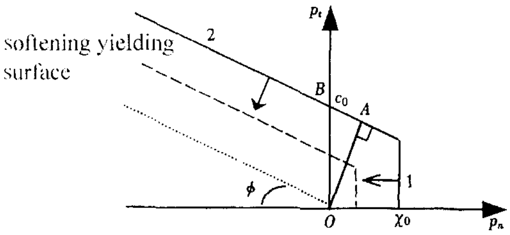

Fig.2 Illustration of yielding surfaces

The coupled relationship between the cohesive laws in both normal and tangential directions of the joint and, further, the softening of the component of $\boldsymbol{Y}$ can be established as follows.

Suppose that, the model of piecewise linear plasticity can be illustrated as Fig.2. Two piecewise linear yielding surfaces are adopted. Then strength vector $\boldsymbol{Y}$ has two components: Y_1 and Y_2. It can be proved that

$$Y_1^0 = \chi_0, \quad Y_2^0 = \frac{c_0}{\sqrt{1+\mu^2}}, \tag{14}$$

where $\mu = \tan\phi$ is the internal frictional coefficient.

It is obvious that the softening of Y_1 is coincide with the softening of $\chi = \chi_0$ corresponding to the evolution of internal variables λ_1, and it can be deduced that

$$Y_1(\boldsymbol{\lambda}) = Y_1^0 + H_n\lambda_1 = \chi_0 + H_n\lambda_1. \tag{15}$$

The softening of Y_2, which is illustrated by OA line in Fig. 2, is the combined result of cohesive softening in both tensile property and shear property of the joint surface crack, i.e., the sum of the projections of softening of $X = X(\boldsymbol{\lambda})$ and $c = c(\boldsymbol{\lambda})$. There is

$$Y_2(\boldsymbol{\lambda}) = Y_2^0 + c(\boldsymbol{\lambda})\cos\phi + \chi(\boldsymbol{\lambda})\sin\phi = \frac{c_0}{\sqrt{1+\mu^2}} + \frac{H_t(\lambda_2+\lambda_3)}{\sqrt{1+\mu^2}} + \frac{\mu H_n \lambda_1}{\sqrt{1+\mu^2}}. \tag{16}$$

The reason of having two internal variables, λ_2 and λ_3, in Eq. (16) for the softening of shear property is that, for shear property, both negative and positive displacement loading can result in the same softening effect, consequently the shrinkage of the elastic domain in traction space is symmetric to the axis of p_n.

Because of the fact that, when macro crack is formed under the action of mode II loading, the tensile strength in normal direction of the joint surface crack will become zero, it is necessary to introduce the influence of internal variables in tangential direction, λ_2 and λ_3, into the softening of the tensile strength parameter. It is assumed in this model that, tensile strength in normal direction of the joint surface crack will also become zero as the tangential strength softening is completed, that is

$$Y_1(\boldsymbol{\lambda}) = Y_1^0 + H_n\lambda_1 + \frac{\chi_0}{c_0}H_t(\lambda_1+\lambda_2) = \chi_0 + H_n\lambda_1 + \frac{\chi_0}{c_0}H_t(\lambda_1+\lambda_2). \tag{17}$$

Equations (16) to (17) are the fundamental relations of strength softening for the piecewise linear plastic yielding surfaces.

1.1.4 Specialised piecewise linear elasto-plastic joint model

There are three pieces of Mohr-Coulomb type linear yielding surface being adopted in this model. The matrix-form of piecewise linear plastic yielding surfaces for any given traction vector $\boldsymbol{p}$ are the same as that described in Eq. (10).

According to the model described in Subsection 2.1.2, those matrices related to Eq. (10) are expressed in specialised form as below. The formulation presented here is limited to 2-dimensional quasi-brittle fracture crack problems, there is

$$\boldsymbol{p} = \begin{Bmatrix} p_n \\ p_t \end{Bmatrix}, \quad \boldsymbol{w} = \begin{Bmatrix} w_n \\ w_t \end{Bmatrix}. \tag{18}$$

The vector of yielding surfaces adopted in this model is

$$\boldsymbol{\varphi} = [\varphi_1 \quad \varphi_2 \quad \varphi_3]. \tag{19}$$

The matrix of collection of normal directions of each yielding surface is

$$\boldsymbol{N} = \begin{bmatrix} 1 & \mu & \mu \\ 0 & 1 & -1 \end{bmatrix}, \tag{20}$$

where $\boldsymbol{N}$ is not unit matrix but only the outward normal direction matrix of yielding surfaces. Relevant changes have been made on relevant terms simultaneously.

According to the softening of strength parameters, the matrix of displacement-softening-modulus is obtained

$$\boldsymbol{H} = \begin{bmatrix} H_n & \frac{\chi_0}{c_0}H_t & \frac{\chi_0}{c_0}H_t \\ \mu H_t & H_t & H_t \\ \mu H_t & H_t & H_t \end{bmatrix}. \tag{21}$$

Vector of plastic multipliers is

$$\boldsymbol{\lambda} = [\lambda_1 \quad \lambda_2 \quad \lambda_3]. \tag{22}$$

The vector of initial strength parameters of both tension and shear behaviour is

$$\boldsymbol{Y} = [\chi_0 \quad c_0 \quad c_0]. \tag{23}$$

In the above equations, material constants and model parameters are

$$\mu, c_0, \chi_0, G_f^{\mathrm{I}}, G_f^{\mathrm{II}}, H_n, H_t. \tag{24}$$

1.1.5 Dilatancy behaviour of the proposed elasto-plastic joint model

Dilatancy means that, plastic displacements in tangential directions will result in plastic displacement in normal direction, and then will result in lateral compression when displacement is constrained in normal direction. This dilatancy-resulted compression will result in increase of loading capacity of the material in shear direction. Usually, as also discussed by some other Refs. [2], [6], dilatancy will vanish under some loading condition. Carol-Prat-Lopez[2] proposed that the dilatancy will asymptotically decrease in the failure process and will vanish at a critical value of compressive traction. In this article, it is alternatively assumed that, dilatancy will vanish as the softening process of shear strength is completed. This assumption is based on the physical background that, at the end of shear-softening-process, the state of complete decohesion will be reached and the total amount of dilatancy will reach the height of asperity at the joint surface, and further amount of dilatancy is meaningless. In order to realised this assumption, non-associated flow law is adopted for the calculation of displacement discontinuity, which is actually only frictional sliding on the crack surfaces. The direction of the plastic flow related to the final Mohr-Coulomb yielding surface is

$$\boldsymbol{V}_2 = \begin{cases} (\mu \quad 1)^{\mathrm{T}} & (\text{if } Y_2(\boldsymbol{\lambda}) > 0), \\ (\mu \quad 0)^{\mathrm{T}} & (\text{if } Y_2(\boldsymbol{\lambda}) \leqslant 0) \end{cases} \tag{25}$$

and that for $\boldsymbol{V}_3$ can be obtained by analogy to Eq. (25).

1.2 Incremental form of piecewise linear elasto-plastic joint model

For the case of complicated external loading conditions, total-quantity-form of plastic theory cannot meet the needs of accuracy requirement. Calculation has to be carried on by a step-by-step method. Consequently incremental formulation becomes necessary.

Even for the incremental case, it is also assumed that all the fundamental relationships for the coupling between fracture process and piecewise linear plasticity expressed in Eqs. (12), (13) and Eqs. (16) – (25) always holds, but the following incremental relationships have to be adopted instead of their counterparts in total-quantity-form formulation, i.e.,

$$\boldsymbol{\varphi} = \boldsymbol{N}^{\mathrm{T}}(\boldsymbol{p}^0 + \Delta\boldsymbol{p}) - \boldsymbol{H}(\boldsymbol{\lambda}^0 + \Delta\boldsymbol{\lambda}) - \boldsymbol{Y}^0 \leqslant \boldsymbol{0}. \tag{26}$$

Equation (26) means that the traction points on the consequent softening yielding surface is loading-path-dependent. And, we also have

$$\Delta \boldsymbol{w} = \Delta \boldsymbol{w}^{e} + \Delta \boldsymbol{w}^{p}. \tag{27}$$

If incremental traction vector $\Delta \boldsymbol{p}$ is given as loading data, there are

$$\Delta \boldsymbol{w}^{e} = \boldsymbol{K}^{-1} \Delta \boldsymbol{p}, \tag{28}$$

$$\Delta \boldsymbol{w}^{p} = \boldsymbol{V} \Delta \boldsymbol{\lambda}. \tag{29}$$

As incremental displacement discontinuity vector $\Delta \boldsymbol{w}$ is given as input data, there is

$$\Delta \boldsymbol{p} = \boldsymbol{K}(\Delta \boldsymbol{w} - \Delta \boldsymbol{w}^{p}) = \boldsymbol{K}(\Delta \boldsymbol{w} - \boldsymbol{V} \Delta \boldsymbol{\lambda}). \tag{30}$$

The displacement-form of Eq.(26) is

$$\boldsymbol{\varphi} = \boldsymbol{N}^{T}(\boldsymbol{w}^{0} + \Delta \boldsymbol{w}) - (\boldsymbol{H} + \boldsymbol{N}^{T}\boldsymbol{K}\boldsymbol{V})(\boldsymbol{\lambda}^{0} + \Delta \boldsymbol{\lambda}) - \boldsymbol{Y}^{0} \leqslant \boldsymbol{0}. \tag{31}$$

Therefore the vector $\boldsymbol{\varphi}$ in Eq. (31) and $\Delta \boldsymbol{\lambda}$ in Eq. (29) has the following complementarity relationship:

$$-\boldsymbol{\varphi} \geqslant \boldsymbol{0}, \quad \Delta \boldsymbol{\lambda} \geqslant \boldsymbol{0}, \quad -\boldsymbol{\varphi}^{T} \cdot \Delta \boldsymbol{\lambda} = \boldsymbol{0}. \tag{32}$$

Solving above nonlinear complementarity problems can then find the traction response for a given incremental displacement loading.

2 Numerical Tests

The models proposed here have been implemented in a set of subroutines of constitutive models for 2D analysis for the purpose of constitutive verification. This subroutine can carry on the calculation of the displacement-to-traction type calculations. The problems are then solved by using the mathematical programming solver, instead of Newton-Raphson type of solver.

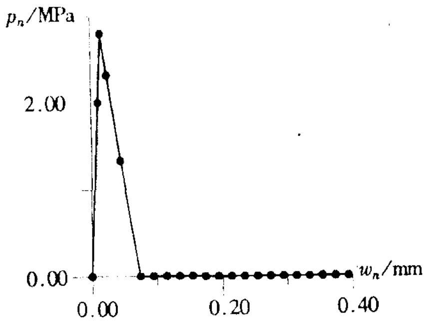

Fig.3 Mode Ⅰ displacement-traction response

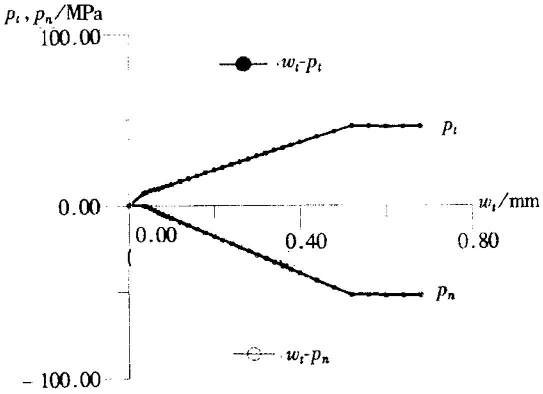

Fig.4 Mode Ⅱ displacement-traction response

The first calculation carried out in this study is to demonstrate the behavior of the model for monotonic displacement loading in the normal direction. Results are shown in Fig. 3. The parameter values used in these calculations are:

$K_n = K_t = 200$ MPa/mm; $\phi = \arctan(0.9)$, $\mu = 0.9$; $\chi_0 = 2.8$ MPa; $c_0 = 7.0$ MPa; $G_f^{\mathrm{I}} = 0.1$ N/mm; $G_f^{\mathrm{II}} = 1.0$ N/mm.

The second calculation is the mode Ⅱ displacement loading, i.e., $w_n = 0$, $w_t =$

$w_t^0 C(t)$, where $C(t)$ is a loading factor. It is shown in Fig. 4 that after the elastic part, dilatancy caused lateral compression continuous increases until the finish of the softening in shear strength. And after that point, only frictional sliding occurs on this interfacial point.

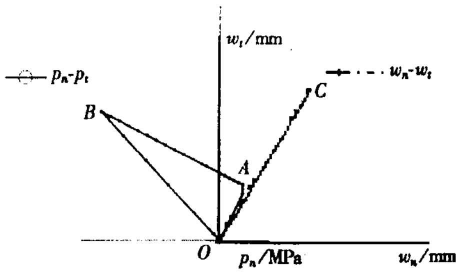

Fig.5 Evolution of traction states and their corresponding displacement states

Figure 5 shows the evolution of the immigration of traction point, i.e., the evolution of the consequent yielding surfaces in traction space corresponding to the given mixed-mode displacement loading. It is indicated in Fig. 5 that, after the first elastic stage, the traction in normal direction decreases because of the compression caused by dilatancy until the point B is reached. Point B marks the end of the fracture process in shear. After point B, the contacted joint surface will eventually become detached under the loading of positive displacement loading in normal direction Ⅱ.

3 Conclusion

This paper presents a piecewise linear elastoplastic constitutive model for the coupled failure process of decohesion and frictional sliding. The coupling between the cohesive fracture and corresponding softening of the macroscopic plasticity is modelled in a natural way by which no more material constant is required except those already adopted in geo-engineering. Numerical results indicate that this model can catch major phenomena existed in the failure process at the joint surface.

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